QUASI-AFFINE SCHEMES AND QUASI-AFFINE MORPHIMS

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CONTENTS

REFERENCE

Most of the materials are extracted from the Stacks Project.

1 QUASI-AFFINE SCHEMES

Definition 1. A scheme is **quasi-affine** if it's quasi-compact and isomorphism to an open subscheme of an affien scheme.

Every quasi-coherent sheaf on a quasi-affine scheme is inherited from the "ambient" affine scheme.

Lemma 1. *Let A be a ring and $U \subset \operatorname{Spec}(A)$ be a quasi-compact open subscheme. Let $\mathscr{F}$ be a quasi-coherent sheaf on U. Then the canonical map*

$$\widetilde{H^0(U,\mathcal{F})}|_U \to \mathcal{F}$$

is an isomorphism.

Proof. Let $j : U \to \operatorname{Spec}(A)$ be the open immersion. Since j is both quasi-coherent and seperated. j_* turns quasi-coherent sheaves to quasi-coherent sheaves. Hence $j_*\mathscr{F}$ is quasi-coherent and thus $j_*\mathscr{F} = \widetilde{H^0(U,\mathcal{F})}$. Restricting it back to U, we can see the canonical morphism is an isomorphism by checking it on stalks. □

We need another lemma before we give a criterion for quasi-affine schemes.

Lemma 2. *Let X be a quasi-compact and quasi-separated scheme. Let $f \in \Gamma(X, \mathscr{O}_X)$. Assume that X_f is affine. Then the canonical morphism*

$$j : X \to \operatorname{Spec}(\Gamma(X, \mathscr{O}_X))$$

induces an isomorphism of $X_f = j^{-1}(D(f))$ onto the standard affine open $D(f) \subset \operatorname{Spec}(\Gamma(X, \mathscr{O}_X))$.

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Proof. It suffice to show that $\mathscr{O}_X(X_f) = \Gamma(X, \mathscr{O}_X)_f$. Let $R = \Gamma(X, \mathscr{O}_X)$. Since j_* is quasi-compact and quasi-separated. We see that $j_*\mathscr{O}_X$ is quasi-coherent on $\mathrm{Spec}(R)$. Hence $j_*\mathscr{O}_X = \widetilde{R}$ and we have

$$\mathscr{O}_X(X_f) = \Gamma(D(f), j_*\mathscr{O}_X) = \Gamma(D(f), \mathscr{O}_{\mathrm{Spec}(R)}) = R_f$$

Hence the lemma holds. □

The we show that a quasi-affine scheme can be canonically embedded into an affine scheme.

Proposition 1. *Let X be a scheme. Then X is quasi-affine iff the canonical morphism*

$$X \to \mathrm{Spec}(\Gamma(X, \mathscr{O}_X))$$

is a quasi-compact open immersion.

Proof. The "if" part is trivial. We only need to show the "only if" part. Suppose that $X \subset \mathrm{Spec}(R)$ is quasi-compact open. Let $A = \Gamma(X, \mathscr{O}_X)$. Consider the ring map $R \to A$ coming from the restriction. Then we have a factorization

$$X \to \mathrm{Spec}(A) \to \mathrm{Spec}(R)$$

Let $x \in X$. Choose $r \in R$ s.t. $x \in D(r) \subset X$. And let $f \in A$ the image of r in A. Hence we have $X_r = \mathrm{Spec}(A)_f = \mathrm{Spec}(R)_r$. And thus $A_f = R_r$. Therefore $D(r) \to \mathrm{Spec}(A)$ is an isomorphism onto $D(f) \subset \mathrm{Spec}(R)$. In conclusion, the proposition holds. □

By the lemma below, the proposition has a equivalent form:

Lemma 3. *Let A be a ring. Then* $\mathrm{Spec}(A) = \mathit{Proj}(A[X])$.

Proposition 2. *Let X be a scheme. Then X is quasi-affine iff $\mathscr{O}_X$ is ample.*

Proof. Suppose X is quasi-affine. Let $A = \Gamma(X, \mathscr{O}_A)$. Consider the open immersion

$$j : X \to \mathrm{Spec}(A) = \mathit{Proj}(A[X])$$

Hence $\mathscr{O}_X$ is ample.

On the other hand, suppose $\mathscr{O}_X$ is ample. Note that $\gamma_*(X, \mathscr{O}_X) \simeq A[X]$ as graded rings. Hence

$$f : X \to \mathit{Proj}(A[X]) = \mathrm{Spec}(A)$$

is an open immersion. □

Corollary 1. *Let X be a quasi-affine scheme. For any quasi-compact immersion $i : X' \to X$, the scheme X' is quasi-affine.*

Proof. Since $\mathscr{O}_X$ is quasi-affine, we know $\mathscr{O}_X$ is ample. Then $\mathscr{O}_{X'}$ is also ample becasue i is a quasi-compact immersion. Hence X' is quasi-affine. □

Let $U \to V$ be an open immersion of quasi-affine schemes. Then their corresponding canonical morphisms in the last proposition have the following relationship:

Lemma 4. *Let $U \to V$ be an open immersion of quasi-affine schemes. Then*

$$\begin{array}{ccc} U & \xrightarrow{\quad j \quad} & \mathrm{Spec}(\Gamma(U, \mathcal{O}_U)) \\ \downarrow & & \downarrow \\ U \longrightarrow V & \xrightarrow{j'} & \mathrm{Spec}(\Gamma(V, \mathcal{O}_V)) \end{array}$$

is cartesian.

Proof. Let $A = \Gamma(U, \mathscr{O}_U)$ and $B = \Gamma(V, \mathscr{O}_V)$. U can be covered by some affine opens of the form $D(g) \subset \operatorname{Spec}(B), g \in B$. Suppose that $D(g)$ is affine and contained in U. Let f be the image of g in A. Then $U_f = V_g$. And thus $V_g \times_{\operatorname{Spec}(R)} \operatorname{Spec}(A)$ is an isomorphism onto $D(f) \subset \operatorname{Spec}(A)$. Since j maps U_f isomorphic to $D(f)$. We have $U_f = U_f \times_{\operatorname{Spec}(B)} \operatorname{Spec}(A)$. Hence globally the diagram is cartesian. □

There is a tricky lemma about quasi-affine scheme, but I still don't know where I will use it. Perhaps I can take it an exercise for my future students?

Lemma 5. *Let X be a quasi-affine scheme. There exists an integer $n \geq 0$, an affine scheme T, and a morphism $T \to X$ s.t. for every morphism $X' \to X$ with X' affine the fibre product $X' \times_X T$ is isomorphic to $\mathbb{A}^n_{X'}$ over X'.*

2 QUASI-AFFINE MORPHISMS

Definition 2. A morphism of schemes $f : X \to S$ is called **quasi-affine** if the inverse image of every affine open of S is a quasi-affine scheme.

Here are some trivial properties of quasi-affine morphisms:

Lemma 6. *A quasi-affine morphism is separated and quasi-compact.*

Proof. Let $f : X \to S$ be quasi-affine. Then it's obviously quasi-compact. On the other hand, we only need to show a morphism from a quasi-affine scheme X to an affine scheme $S = \operatorname{Spec}(A)$ is quasi-compact. We have the following factorization:

$$X \to \operatorname{Spec}(\Gamma(X, \mathscr{O}_X)) \to \operatorname{Spec}(A)$$

□

Lemma 7. *A quasi-compact immersion is quasi-affine.*

Proof. Let $f : X \to Z$ be a quasi-compact immersion. WLOG we assume that Z is affine. By the lemma in the previous section, we know that X is quasi-affine. □

Lemma 8. *Let S be a scheme and X be an affine scheme. A morphism $f : X \to S$ is quasi-affine iff it's quasi-compact. In particular any morphism form an affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. The first assertion is trivial. And consider

$$X \to S \to \operatorname{Spec}(\mathbb{Z})$$

Since $X \to \operatorname{Spec}(\mathbb{Z})$ is quasi-compact and $S \to \operatorname{Spec}(\mathbb{Z})$ is quasi-separated, $X \to S$ is quasi-compact. □

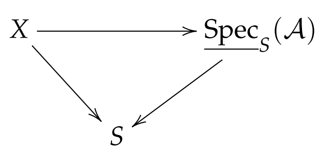
Then we use the relative spectrum to depict quasi-affine morphisms. For more about relative spectrum, see the Stacks Project.

Proposition 3. *Let $f : X \to S$ be a morphism of schemes. TFAE*

(1) f is quasi-affine

(2) There exists an affine open covering $S = \bigcup W_j$ s.t. $f^{-1}(W_j)$ is quasi-affine.

(3) There exists a quasi-coherent sheaf of $\mathscr{O}_S$-algebra $\mathcal{A}$ and a quasi-compact open immersion over S:



(4) *Same as in (3) but with $\mathcal{A} = f_*\mathscr{O}_X$ and the horizontal arrow is the canonical morphism s.t. for each affine open $U \subset S$, the restriction of the morphism on $f^{-1}(U)$ is the canonical morphism*

$$f^{-1}(U) \to \operatorname{Spec}(\Gamma(f^{-1}(U), \mathscr{O}_X))$$

Proof. Obviously we have (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3).

Then we show that (3) $\Rightarrow$ (1). Let g be the horizon morphism and h be the third morphism in the triangle. By the construction of the relative spectrum, we have $h^{-1}(U)$ affine for any affine open $U \subset S$. Hence $f^{-1}(U) = g^{-1}h^{-1}(U)$ is an quasi-compact open subscheme of $h^{-1}(U)$. And thus f is quasi-affine.

At last, we show that (2) $\Rightarrow$ (4). Obviously we have f quasi-compact and quasi-separated. Hence the relative spectrum and the canonical morphism exist. Since for each affine open W_j, the canonical morphism

$$f^{-1}(W_j) \to \operatorname{Spec}(\Gamma(f_*\mathscr{O}_X, W_j)) = \operatorname{Spec}(\Gamma(\mathscr{O}_X, f^{-1}(W_j)))$$

is an open immersion because $f^{-1}(W_j)$ is quasi-affine. Hence (4) holds. □

By this proposition, we can show a lot of conclusions about quasi-affine morphisms hold.

Lemma 9. *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quasi-affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is quasi-affine. Let $j : g^{-1}(U) \to V$ be a quasi-compact open immersion to an affine scheme V.

By the previous proposition, $f^{-1}g^{-1}(U)$ is a quasi-compact open subscheme of $\underline{\operatorname{Spec}}_{g^{-1}(U)}(\mathcal{A})$ for some quasi-coherent sheaf of $\mathscr{O}_{g^{-1}(U)}$-algebras $\mathcal{A}$. And $\mathcal{A}' = j_*\mathcal{A}$ is a quasi-coherent sheaf s.t. $j^*\mathcal{A}' = \mathcal{A}$. Hence we have a commutative diagram

$$\begin{array}{ccccc} f^{-1}(g^{-1}(U)) & \longrightarrow & \underline{\operatorname{Spec}}_{g^{-1}(U)}(\mathcal{A}) & \longrightarrow & \underline{\operatorname{Spec}}_V(\mathcal{A}') \\ & & \downarrow & & \downarrow \\ & & g^{-1}(U) & \xrightarrow{\quad j \quad} & V \end{array}$$

where the square is a fibre square. Note that the upper right morphism is an open immersion and the upper right corner is an affine scheme. Hence $(g \circ f)^{-1}(U)$ is quasi-affine. □

Lemma 10. *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Let $f : X \to S$ be a quasi-affine morphism and $g : S' \to S$ be a morphism. We can find a quasi-coherent sheaf of $\mathscr{O}_S$-algebra $\mathcal{A}$ and a quasi-compact open immersion $X \to \underline{\operatorname{Spec}}_S(\mathcal{A})$. Let $f' : X_{S'} \to S'$ be the base change of f. Sicen the base change of a quasi-compact open immersion is still a quasi-compact open immersion, we see that $X_{S'} \to \underline{\operatorname{Spec}}_{S'}(g^*\mathcal{A}) = S' \times_S \underline{\operatorname{Spec}}_S(\mathcal{A})$. Hence we conclude that f' is quasi-affine. □

Lemma 11. *Suppose that $g : X \to Y$ is a morphism of schemes over S. If X is quasi-affine over S and Y is quasi-separated over S, then g is also quasi-affine. In particular, any morphism from a quasi-affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. The base change $X \times_S Y \to Y$ is quasi-affine. The morphism $X \to X \times_S Y$ is a quasi-compact immersion since $Y \to S$ is quasi-separated. A quasi-compact immersion is quasi-affine and the composition of quasi-affine morphisms is quasi-affine. □