NORMALITY

XUANDE LIU*

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CONTENTS

1 NORMAL DOMAINS

Definition 1 (Normal domains)**.** Let A be an integral domain and $K = \mathrm{Frac}(A)$. If A is integral closed in K then we say A is **normal**. For instance, a principal ideal domain is normal.

The localizations of normal integral domains are also normal, and an integral domain is normal iff all of its localizations are normal.

Proposition 1. *Let A be an integral domain. TFAE*

1. *A is normal*
2. *$A_{\mathfrak{p}}$ is normal for every prime $\mathfrak{p}$*
3. *$A_{\mathfrak{m}}$ is normal for every maximal prime $\mathfrak{m}$*

Proof. $1 \Rightarrow 2$: Suppose we have

$$a^n + \sum_{i=0}^{n-1} b_i a^i = 0$$

* *School of Mathematical Sciences, Peking University, Beijing, China*

where $a \in K = \mathrm{Frac}(a)$ and $b_i \in A_\mathfrak{p}$. Then we can find $s \in A \backslash \mathfrak{p}$ s.t. $sb_i \in A$ for all i. Then

$$(sa)^n + \sum_{i=0}^{n-1} b_i \cdot s^{n-i}(sa)^i = 0$$

Hence we have $sa \in A$, hence $a \in A_\mathfrak{p}$.

$2 \Rightarrow 3$:obvious. $3 \Rightarrow 1$: Notice that the arbitrary intersection of integral closed domains are also integral closed. And we have

$$A = \bigcap_{\mathfrak{m} \text{ max}} A_\mathfrak{m}$$

□

We now introduce a conception close to integral.

Definition 2 (Almost integral)**.** Let A be an integral domain and $K = \mathrm{Frac}(A)$. We say that $k \in K$ is **almost integral** over A if there exists $a \in A$ s.t. $ak^n \in A$ for all a.

Proposition 2. *If k is integral over A then it's also almost integral over A. If A is Noetherian then the converse also holds.*

Proof. Suppose we have

$$k^n + \sum_{i=0}^{n-1} a_i k^i = 0$$

where $a_i \in A$. We can find $b \in A$ s.t. $bk^i \in A$ for all $i \leqslant n$. By induction we can show that b is exactly the element we want.

If A is Noetherian, then $A[k]$ is a submodule of $a^{-1}A$, hence is f.g. Therefore k is integral over A. □

Definition 3 (Completely Normal)**.** We say that an integral domain A is **completely normal** if every element $k \in K = \mathrm{Frac}(A)$ which is almost integral over A belongs to A.

In order to show that $A[X]$ is normal (resp. completely normal) if A is normal (resp. cpt normal), we need the following lemma.

Lemma 1. *Let A be a domain and $K = \mathrm{Frac}(A)$. Suppose $f = \sum \alpha_i x_i \in K[X]$.*

1. *If f is almost integral over $A[X]$, the all α_i are a.i. over A*
2. *If f is integral over $A[X]$, then all α_i are integral over A*

Proof. 1. Suppose that there exists $g = \sum \beta_i x^i \in A[X]$ s.t. $gf^i \in A[X]$ for all $i > 0$. Then $\alpha_n^i \beta_n \in A$ for all $i > 0$. Hence α_n is a.i. over A. Hence $f - \alpha_n x^n$ is a.i. over A and by induction we win.

2. Suppose $\alpha_i = a_i/b_i$. Suppose $P(t) = t^d + \sum_{i<d} f_i t^i$ s.t. $P(f) = 0$ and $f_i \in A[X]$. Let A_0 be the subring generated by $1, b_i, a_i$ and all the coefficients of those f_i. Then A_0 is the image of $\mathbb{Z}[X_1, \cdots, X_m]$ for some m and thus Noetherian. Since $f \in \mathrm{Frac}(A_0[X])$ is integral over $A_0[X]$, it's also a.i. over $A_0[X]$ and hence all the α_i are a.i. over A_0, which means α_i are integral over A_0 since A_0 is Noetherian.

□

Then we can show that normality and cpt normality are stable under taking polynomial rings.

Proposition 3. *1. Let A be a completely normal domain. Then both $A[X]$ and $A[[X]]$ are completely normal.*

2. Let A be a normal domain. Then $A[X]$ is a normal domain.

3. Let A be a Noetherian normal domain. Then $A[[X]]$ is a Noetherian normal domain.

Proof. Let $K = \mathrm{Frac}(A)$, then $K[X]$ is a PID. Hence it's normal and completely normal. And a element in $\mathrm{Frac}(A[X])$ that is integral or a.i. over $A[X]$ should lie in $K[X]$. Now by the previous lemma we know that all the coefficients of this element are contained in A. Thus $A[X]$ is normal (resp. cpt normal) if A is normal (resp. cpt normal).

In the same way we can show that $f \in K[[X]]$ is a.i. over A iff all the coefficients are a.i. over A. And since $K[[X]]$ is also a PID, the conclusion holds. As for the last assertion, it's a direct corollary of the cpt normal case. □

Remark. There exists a normal ring A s.t. $A[[X]]$ is not normal. Here is a counterexample.

2 NORMAL RINGS

Definition 4 (Normal rings)**.** A ring R is called **normal** if for every prime $\mathfrak{p} \subset R$ the localization $R_\mathfrak{p}$ is a normal domain.

Proposition 4. *A normal ring must be reduced, because the ideal of all the nilpotent elements has empty support.*

Proposition 5. *Let R be a normal ring, Then $R[X]$ is also a normal ring. Let $P \subset R[X]$ be a prime and $\mathfrak{p} = P \cap R$. Hence $R[X]_P$ is a localization of $R_\mathfrak{p}[X]$, hence it is a normal domain.*

Proposition 6. *If a ring R is normal, then it's integrally closed in its total ring of fractions $Q(R)$.*

Proof. Let R be a normal ring. Suppose $u \in Q(R)$ is integral over R. Let $I = \{x \in R | xu \in R\}$. It suffice to show I is not contained in any prime ideal of R. Let $\mathfrak{p} \subset R$ be a prime. Since $R \to R_\mathfrak{p}$ is flat, tensor $R \to Q(R)$ with $R_\mathfrak{p}$ we have $R_\mathfrak{p} \subset Q(R) \otimes R_\mathfrak{p}$. Since $R_\mathfrak{p}$ is a normal domain, $u \times 1 \in R_\mathfrak{p}$. Suppose $u \times 1 = a \otimes 1/f$ for $a \in R$ and $f \in R \backslash \mathfrak{p}$. Then $fu - a$ maps to zero in $Q(R)_\mathfrak{p}$. Hence there exists $f' \in R \backslash \mathfrak{p}$ s.t. $f'fu = f'a \in R$. Hence $ff' \in I$. Since $f, f' \notin \mathfrak{p}$, $ff' \notin \mathfrak{p}$ and $I \subsetneq \mathfrak{p}$. □

Then we begin to explore the structure of normal rings.

Proposition 7. *A finite product of normal rings is normal.*

Proof. Let R, S be normal rings. Every primes of $R \times S$ are of the form $R \times \mathfrak{p}$ or $\mathfrak{q} \times S$. Hence we have $(R \times S)_{\mathfrak{q} \times S} = R_{\mathfrak{q}}$,which is a normal domain. The other condition is also the same. □

We need some results about total ring of fractions of a reduced ring before the structure theorem.

Lemma 2. 1. *If R is a reduced ring. Then*

   a) *if $\mathfrak{p}$ is a minimal prime of R, then $R_{\mathfrak{p}}$ is a field.*

   b) *R is a subring of product of fields.*

   c) *$R \to \prod_{\mathfrak{p}\ minimal} R_{\mathfrak{p}}$ is an embedding into the product of fields.*

   d) *$\cup_{\mathfrak{p}\ minimal}\, \mathfrak{p}$ is the set of zerodivisors of R.*

2. *Let R be a ring. Assume that R has finitely many minimal primes $\mathfrak{q}_1, \cdots, \mathfrak{q}_t$, and that $\cup \mathfrak{q}_i$ is the set of zerodivisors of R. Then the total ring of fractions $Q(R)$ is equal to $R_{\mathfrak{q}_1} \times \cdots \times R_{\mathfrak{q}_t}$.*

Proof. 1. a) Every elements of $\mathfrak{p}R_{\mathfrak{p}}$ is nilpotent, and hence $\mathfrak{p}R_{\mathfrak{p}} = 0$.

   b) The kernel of $R \to R_{\mathfrak{p}}$ is $\mathfrak{p}$, and we have $\cap \mathfrak{p} = 0$. Hence we conclude both (b) and (c).

   c) As above.

   d) If $xy = 0$ and $y \neq 0$, then $y \neq \mathfrak{p}$ for some minimal prime $\mathfrak{p}$. Hence $x \in \mathfrak{p}$. Thus if y not contained in any minimal primes we have $x = 0$. On the other hand, since $\mathfrak{p}R_{\mathfrak{p}} = 0$ for all minimal prime $\mathfrak{p}$, every element in $\mathfrak{p}$ is zerodivisor. (Notice that we don't have the hypothesis of being Noetherian here.)

2. There is a natural maps $Q(R) \to R_{\mathfrak{q}_i}$ since any nonzerodivisor is contained in $R \backslash \mathfrak{q}_i$. Hence there exists $Q(R) \to R_{\mathfrak{q}_1} \times \cdots \times R_{\mathfrak{q}_t}$. For any nonminimal prime $\mathfrak{p} \subset R$, it must not contained in $\cup \mathfrak{q}_i$. Thus $\mathfrak{p}$ must have a nonzerodivisor. Hence $\mathrm{Spec}(Q(R)) = \{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$. Therefore $\mathrm{Spec}(Q(R))$ is a finite discrete set and it follows that $Q(R) = A_1 \times \cdots \times A_t$ with $\mathrm{Spec}(A_i) = \{\mathfrak{q}_i\}$. Moreover A_i is a local ring, which is a localization of $Q(R)$ and hence R. It's also a subset of $R_{\mathfrak{q}_i}$. Hence $A_i \simeq R_{\mathfrak{q}_i}$.

□

Theorem 1 (The structure of normal rings). *Let R be a reduced ring with finitely many minimal primes(for instance when R is Noetherian). TFAE*

1. *R is normal*

2. *R is integrally closed in its total ring of fractions*

3. *R is a finite product of normal domains*

Proof. Obviously we have $1 \Rightarrow 2$ and $3 \Rightarrow 1$. It suffice to show $2 \Rightarrow 3$. By the lemma we have $Q(R) \simeq \prod_{\mathfrak{p} \text{ minimal}} R_{\mathfrak{p}}$. If R is integrally closed, then it must contain every $e_i = (0, \cdots, 1, \cdots, 0)$. Hence it's a product of domains. Every factor of the form $R/\mathfrak{q}$ with fraction of field $R_{\mathfrak{p}}$. Hence by the lemma below, all the map $R/\mathfrak{q} \to R_{\mathfrak{p}}$ is integrally closed. And the assertion holds. □

Lemma 3. *Let $R = \prod R_i$ and $S = \prod S_i$ and $R_i \to S_i$ be ring morphisms. $s = (s_i) \in S$ is integral over R iff each s_i is integral over R_i.*

At the last of this section, we will show the going down theorem holds for integral extension over normal rings. But at first we need some technical preparation. For the details of the proof, here is the reference.

Definition 5 (Integral over an ideal). Let $\phi : R \to S$ and $I \subset R$ be an ideal. We say $s \in S$ is **integral over** I if there exists $P = x^d + \sum_{i<d} a_i x^i$ with coefficients $a_i \in I^{d-i}$ s.t. $P^{\phi}(g)0$ on S.

We give some characterization of the elements that integral over I.

Lemma 4. *Let $\phi : R \to S$ and $I \subset R$ be an ideal.*

1. *Let $A = \sum I^n t^t \subset R[t]$ be the subring of polynomial ring generated by $R \oplus It$. $s \in S$ is integral over I iff $st \in S[t]$ is integral over A.*
2. *The set of elements of S that are integral over I form a R-submodule of S. Furthermore, if $s \in S$ in integral over R and s' integral over I, then ss' is integral over I.*
3. *If ϕ is integral. Then every element of IS is integral over I.*

Lemma 5. *Let K be a field. And $a_0, \cdots, a_{n-1}, b_0, \cdots, b_{m-1} \in K$. If the polynomial $x^n + \sum_{i<n} a_i x^i$ divides $x^m + \sum_{j<m} b_j x^j$ in $K[X]$. Then*

1. *$a_0, \cdots, a_{n-1}$ are integral over any subring R_0 of K containing the elements $b_0, \cdots, b_{m-1}$*
2. *each a_i lies in $\sqrt{(b_0, \cdots, b_{m-1})R}$ for any subring $R \subset K$ containing the elements $a_0, \cdots, a_{n-1}$ and $b_0, \cdots, b_{m-1}$.*

Proposition 8. *Let $R \subset S$ be an inclusion of domains. Assume R is normal. Let $g \in S$ be integral over R. Then the minimal polynomial of g has coefficients in R.*

Now we conclude the going down theorem holds.

Theorem 2 (Going down for integral over normal rings). *Let $R \subset S$ be an inclusion of domains. Assume R is normal and S integral over R. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}'$ be a prime of S with $\mathfrak{p}' = R \cap \mathfrak{q}'$. Then there exists a prime $\mathfrak{q}$ with $\mathfrak{q} \subset \mathfrak{q}'$ such that $\mathfrak{p} = R \cap \mathfrak{q}$.*

Proof. Notice that if $\mathfrak{p} = \phi^{-1}(\mathfrak{p}S)$, then we have injection $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \to S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$, and hence $S \otimes \kappa(\mathfrak{p}) \neq 0$, thus $\mathfrak{p}$ is in the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ and the conclusion holds. Hence it suffice show that $\mathfrak{p} = \phi^{-1}(\mathfrak{p}S_{\mathfrak{q}'})$.

Let $z = y/g \in \mathfrak{p}S_{\mathfrak{q}'} \cap R$, where $y \in \mathfrak{p}S$ and $g \in S \backslash \mathfrak{q}'$. Since $\mathfrak{p}S$ is integral over $\mathfrak{p}$ by the lemma, there exists a monic polynomial $P = x^m + \sum_{i<m} b_i x^i$ with $b_i \in \mathfrak{p}$ s.t. $P(y) = 0$. Also by the lemma, the minimal polynomial of g over K has coefficients in R. Write it as $Q = x^n + \sum_{j<n} a_i x^i$. Note that not all $a_i \in \mathfrak{p}$, otherwise $g^n \in \mathfrak{q}'$.

Since $y = zg$ we see that $Q' = x^n + za_{n-1}x^{n-1} + \cdots + z^n a_0$ is the minimal polynomial for y. Hence Q' divides P and we see that $z^j a_{n-1} \in \sqrt{(b_0, \cdots, b_{m-1})} \subset \mathfrak{p}$ for all $j = 1, \cdots, n$. Because not all $a_i \in \mathfrak{p}$, we conclude $z \in \mathfrak{p}$ as desired.

□

3 REGULAR LOCAL RINGS

Definition 6 (Orders and Leading Terms)**.** Let A be a ring and I be an ideal with $\cap I^n = (0)$. Then for each $a \neq 0 \in A$, we define **the order of** a to be the maximal integral s.t. $a \in I^n$ and $a \notin I^{n+1}$. We have $\mathrm{ord}(a+b) \geqslant \min(\mathrm{ord}(a), \mathrm{ord}(b))$ and $\mathrm{ord}(ab) \geqslant \mathrm{ord}(a) + \mathrm{ord}(b))$. Let $A' = \mathrm{gt}^I(A) = \oplus I^n/I^{n+1}$. We define **the leading term of** a to be the imagine a^* of a in I^n/I^{n+1}.

Let Q be an ideal of A, the Q^*c is a graded ideal of A^*. We define $\overline{A} = A/Q$ and $\overline{I} = I + Q/Q$. Then we have $\mathrm{gr}^{\overline{I}}(\overline{A}) = \mathrm{gr}^I(A)/Q^*$.

Theorem 3 (Krull)**.** *Let A, I, A' be as above. Then*

1. *If A' is a domain, so is A.*
2. *Suppose that A is Noetherian and $I \subset \mathrm{rad}(A)$. Then if A' is a normal domain, so is A.*

Remark. It can happen that A is a normal domain while A' is not a domain.

Proposition 9. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. Then A is regular iff $\mathrm{gr}(A)$ is isomorphic to the polynomial ring $k[X_1, \cdots, X_d]$.*

Proof. On one hand, if $\mathrm{gr}(A)$ is isomorphic to $k[X_1, \cdots, X_d]$. Then we have $\mathrm{rank}_k \mathfrak{m}/\mathfrak{m}^2 = d$ and $\dim A = d$. Hence it's regular.

On the other hand, let $k[X_1, \cdots, X_d] \to \mathrm{gr}(A)$ be surjective with kernel I, which is a homogeneous ideal. Suppose $f \in I$ is a homogeneous polynomial of deg n_0. Then for n large enough we have $\ell(A/\mathfrak{m}^{n+1}) \leqslant \binom{n+d}{d} - \binom{n+d-n_0}{d}$, hence it is a polynomial of deg less than $d - 1$. Thus it contradicts our assumption. □

Proposition 10. *Let $(A, \mathfrak{m})$ be a regular local ring and $x_1, \cdots, x_d$ a regular system of parameters. Then*

1. *A is a normal domain*

2. $(x_1, \cdots, x_i) = \mathfrak{p}_i$ *is a prime ideal of height* i *for each* $i \leqslant d$*, and* $A/\mathfrak{p}_i$ *is a regular local ring of dimension* $d-i$.

3. $x_1, \cdots, x_d$ *is an* A*-regular sequence. Hence* A *is a Cohen-Macaulay local ring.*

4. *If* $\mathfrak{p}$ *is an ideal of* A *and* $A/\mathfrak{p}$ *is regular with dimension* $d-i$*, then there exists a regular system of parameters* $y_1, \cdots, y_d$ *s.t.* $\mathfrak{p} = y_1, \cdots, y_i$.

Proof. 1. It's the corollary by the Krull's theorem and the proposition above.

2. We have $\dim A/\mathfrak{p}_i = d-i$. Therefore $A/\mathfrak{p}_i$ is regular and hence by (1) we know $\mathfrak{p}_i$ is a prime.

3. It is a direct conclusion by 2.

4. We have $d-i = \mathrm{rank}_k \mathfrak{m}/\mathfrak{m}^2 - \mathrm{rank}_k \mathfrak{m}^2 + \mathfrak{p}/\mathfrak{m}^2$, hence $i = \mathrm{rank}_k \mathfrak{m}^2 + \mathfrak{p}/\mathfrak{m}^2$. We can choose i elements $y_1, \cdots, y_i \in \mathfrak{p}$ that generated $\mathfrak{m}^2 + \mathfrak{p}/\mathfrak{m}^2$. And $d-i$ elements $y_{i+1}, \cdots, y_d$ in $\mathfrak{m}$ s.t. $y_1, \cdots, y_d$ generated $\mathfrak{m}$. Thus we get a system of parameters. Since $(y_1, \cdots, y_i) \subset \mathfrak{p}$ and both of them have dimension $d-i$. We conclude $\mathfrak{p} = (y_1, \cdots, y_i)$.

□

3.1 Regular local ring of dimension 1

Proposition 11. *A regular local ring of dimension* 1 *is a discrete valuation ring, and vice versa.*

Remark. The only Noetherian valuation rings are discrete valuation rings.

We still have another characterization of regular local ring of dimension 1. Namely a Noetherian local ring of dimension 1 is regular iff it's normal. But we need to show a few lemmas before the proof.

Lemma 6. 1. *Let* A *be a Noetherian domain and* $K = \mathrm{Frac}(A)$*. For any ideal* $I \neq 0$*, we define* $I^{-1} = \{x \in K | xI \in A\}$*. Let* $0 \neq a \in A$ *and* $\mathfrak{p} \in \mathrm{Ass}_A(A/aA)$*, then* $\mathfrak{p}^{-1} \neq A$.

2. *Let* $(A, \mathfrak{p})$ *be a Noetherian local domain s.t.* $\mathfrak{p} \neq 0$ *and* $\mathfrak{p}\mathfrak{p}^{-1} = A$*. Then* $\mathfrak{p}$ *is a principal ideal, thus* A *is regular of dimension* 1.

Proof. 1. There exists $b \in A$ s.t. $(aA : b) = \mathfrak{p}$. Hence $(b/a_{\mathfrak{p}} \subset A$ and $b/a \notin A$.

2. Since $\cap \mathfrak{p}^n = 0$, we have $\mathfrak{p} \neq \mathfrak{p}^2$. Let $a \in \mathfrak{p} - \mathfrak{p}^2$. Then $a\mathfrak{p}^{-1} \subset A$ and if $a\mathfrak{p}^{-1} \subset \mathfrak{p}$, then $aA = a\mathfrak{p}^{-1}\mathfrak{p} \subset \mathfrak{p}^2$, contradicting the assumption. Thus $a\mathfrak{p}^{-1} = A$. Hence $aA = \mathfrak{p}$.

□

Theorem 4. *Let* $(A, \mathfrak{p})$ *be a Noetherian local ring of dimension* 1*. Then* A *is regular iff it's normal.*

Proof. Only if: Trivial.

If: It suffice to show $\mathfrak{p}\mathfrak{p}^{-1} = A$. Suppose that $\mathfrak{p}\mathfrak{p}^{-1} = \mathfrak{p}$, then $\mathfrak{p}(\mathfrak{p}^{-1})^n = \mathfrak{p} \subset A$ for any $n > 0$. Hence all the elements of $\mathfrak{p}^{-1}$ are almose integral over A, hence integral over A since A is Noetherian. Therefore we have $\mathfrak{p}^{-1} = A$. Since $\dim A = 1$, we have $\mathfrak{p} \in \mathrm{Ass}(A/aA)$ for any $a \neq 0$. Hence $\mathfrak{p}^{-1} \neq A$ and we win. □

Corollary 1. *Let A be a Noetherian normal domain. Then any non-zero principal ideal is unmixed, and we have*

$$A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$$

If $\dim A \leqslant 2$ *then A is C.M.*

Proof. Let $a \in A$ and $\mathfrak{p} \in \mathrm{Ass}(A/aA)$. WLOG we assume that $(A.\mathfrak{p})$ is local. Then $\mathfrak{p}^{-1} \neq A$. If $\mathrm{ht}(\mathfrak{p}) \geqslant 1$ then ???????????? □

3.2 Serre's conditions

Definition 7 (Serre's Conditions)**.** Let A be a Noetherian ring, we define the following conditions:

(S_k) for any prime $\mathfrak{p}$ we have $\mathrm{depth}_{A_{\mathfrak{p}}} A_{\mathfrak{p}} \geqslant \min\{k, \mathrm{ht}(\mathfrak{p})\}$.

(R_k) for any prime $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) \leqslant k$, then $A_{\mathfrak{p}}$ is regular.

We list a few conclusions about Serre's conditions:

Proposition 12.

1. *A satisfies* S_1 *iff* $\mathrm{Ass}(A)$ *has no embedded primes*
2. *A satisfies* S_2 *iff both* $\mathrm{Ass}(A)$ *and* $\mathrm{Ass}(A/fA)$ *for any nonzerodivisor* $f \in A$ *have no embedded primes*
3. *A satisfies* S_k *for every* k *iff it's C.M.*
4. *A satisfies* S_1 *and* R_0 *iff it's reduced*
5. *A satisfies* S_2 *and* R_1 *iff it's normal*

Proof.

1. Suppose $\mathrm{Ass}(A)$ has no embedded primes, then any prime $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) \geqslant 1$ has at least one nonzerodivisor of A. Otherwise it should be contained in some minimal prime. Thus the condition S_1 holds. Conversely if S_1 holds, then $\mathfrak{p}A_{\mathfrak{p}}$ has at least one nonzerodivisor for any prime $\mathfrak{p}$ with height no smaller than 1. Therefore we have $\mathfrak{p}A_{\mathfrak{p}} \notin \mathrm{Ass}(A_{\mathfrak{p}})$ and hence $\mathfrak{p} \notin \mathrm{Ass}(A)$. Thus $\mathrm{Ass}(A)$ has no embedded prime.
2. Notice that any prime $\mathfrak{p}$ in A/fA corresponds to a prime $\mathfrak{p}'$ with $\mathrm{ht}(\mathfrak{p}') = \mathrm{ht}(\mathfrak{p}) + 1$.
3. Trivial.

4. If A is reduced, then $A_\mathfrak{p}$ is a field for every minimal prime $\mathfrak{p}$. And we have

$$\bigcup_{\mathfrak{p} \text{ minimal}} \mathfrak{p} = \text{The zerodivisors of } A$$

   Therefore A has no embedded primes.

   Conversely, if A satiesfies S_1 and R_0, then we show $A_\mathfrak{p}$ is reduced by induction on $\mathrm{ht}(\mathfrak{p})$.

5. On one hand, if A is normal. Then for every prime $\mathfrak{p}$ with height one, $A_\mathfrak{p}$ is normal, hence it's regular. For prime with height no greater than 2, $A_\mathfrak{p}$ is a normal domain with $\dim A_\mathfrak{p} \leqslant 2$. Tence it's C.M. and we win.

   On the other hand, if A satisfies S_2 and R_0. Then it's reduced, and we have

$$Q(A) = \prod_{\mathfrak{p} \text{ minimal}} A_\mathfrak{p}$$

   It suffice to show A is integrally closed in $Q(A)$. Suppose we have

$$(a/b)^n + \sum c_i (a/b)^i = 0$$

   with $a, b, c_i \in A$ and b is a regular element. Then we have

$$a^n + \sum c_i b^{n-1} a^i = 0$$

   It suffice to show that $a \in bA$. Since bA is unmixed of height 1 by S_2. It suffice to show for any $\mathfrak{p}$ of height 1 we have $a_\mathfrak{p} \in b_\mathfrak{p}A$.(Because A/bA is reduced.) Since $A_\mathfrak{p}$ is regular, and hence normal. Therefore the conclusion holds.

□

Proposition 13. *Let A be a ring s.t. $A_\mathfrak{p}$ is regular for every $\mathfrak{p} \in \mathrm{Spec}(A)$. Then all the local rings of $A[X_1, \cdots, X_n]$ are also regular.*

Proof. It suffice to show the case $n = 1$ and $(A, \mathfrak{p}, k)$ is a regular local ring. Let $B = A[X]$. Suppose $\mathfrak{q} \in \mathrm{Spec}(B)$ and $\mathfrak{p} = \mathfrak{q} \cap A$. We show that $B_\mathfrak{q}$ is regular. Since $B/\mathfrak{p}B = k[X]$. We have $\mathfrak{q} = \mathfrak{p}$ or $\mathfrak{q} = \mathfrak{p} + f(X)B$, where $f(X)$ is a monic polynomial. Let $\dim A = d$ and $\mathfrak{p}$ is generated by d elements. Then $\mathfrak{q}$ is generated by d or $d+1$ elements. But obviously we have $\mathrm{ht}(\mathfrak{p}B) \geqslant d$ in the first condition and $\mathrm{ht}(\mathfrak{q}) \geqslant d+1$ in the latter case. Hence $B_\mathfrak{q}$ is regular and the conclusion holds. □