

# SCHEME THEORETIC NOTIONS

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## REFERENCE

Most of the materials are extracted from the [The Stacks Project](#).

## 1 SCHEME THEORETIC INVERSE IMAGE

**Definition 1.** Let  $f : X \rightarrow Y$  be a morphism. Let  $Z \subset Y$  be a closed or open subscheme of  $Y$ . The inverse image  $f^{-1}(Z)$  is the closed or open subscheme  $Z \times_Y X$  of  $X$ .

## 2 SCHEME THEORETIC INTERSECTION AND UNION

**Definition 2** (Scheme Theoretic Intersection and Union). Let  $X$  be a scheme and  $Y, Z$  are closed subschemes corresponding to quasi-coherent ideal sheaves  $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$ . The **scheme theoretic intersection** (S.T.I in brief) of  $Y$  and  $Z$  is the closed subscheme corresponding to the ideal sheaf  $\mathcal{I} + \mathcal{J}$ . The **scheme theoretic union** (S.T.U in brief) of  $Y$  and  $Z$  is the closed subscheme corresponding to the ideal sheaf  $\mathcal{I} \cap \mathcal{J}$ .

Then we discuss the properties of scheme theoretic intersection and scheme theoretic union.

**Lemma 1.** Let  $X, Y, Z$  be as previous. Let  $Y \cap Z$  be the S.T.I. of  $Y$  and  $Z$ . Then  $Y \cap Z \rightarrow Y$  and  $Y \cap Z \rightarrow Z$  are closed immersions and the following diagram is a cartesian diagram:

$$\begin{array}{ccc} Y \cap Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

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*Proof.* The fact that morphisms are closed immersions are obvious. Let  $U = \text{Spec}(A)$  be an affine open and let  $Y \cap U$  and  $Z \cap U$  correspond to the ideals  $I, J$ . Then we have  $A/I \otimes_A A/J = A/(I+J)$ . Hence by the construction of fibre product we draw the conclusion.  $\square$

**Lemma 2.** Let  $X, Y, Z$  be as previous. Let  $Y \cap Z$  be the S.T.I of  $Y$  and  $Z$  and  $Y \cup Z$  be the S.T.U of  $Y$  and  $Z$ . Then  $Y \rightarrow Y \cup Z$  and  $Z \rightarrow Y \cup Z$  are closed immersions. There exists a short exact sequence of  $\mathcal{O}_X$  modules:

$$0 \rightarrow \mathcal{O}_{Y \cup Z} \rightarrow \mathcal{O}_Y \times \mathcal{O}_Z \rightarrow \mathcal{O}_{Y \cap Z} \rightarrow 0$$

and the following diagram is cocartesian:

$$\begin{array}{ccc} Y \cap Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \cup Z \end{array}$$

*Proof.* The exact sequence comes from the following exact sequence of  $A$ -modules:

$$0 \rightarrow A/I \cap J \rightarrow A/I \times A/J \rightarrow A/(I+J) \rightarrow 0$$

Given morphisms of schemes  $f : Y \rightarrow T$  and  $Z \rightarrow T$  agreeing as  $Y \cap Z \rightarrow T$ . We need to construct the unique morphism  $h : Y \cup Z \rightarrow T$ . Suppose  $x \in Y \setminus Z$ . Then  $Y \rightarrow Y \cup Z$  is an isomorphism in some neighborhood of  $x$  and  $h$  can be uniquely defined on such a neighborhood. If  $x \in Y \cap Z$ . Then there exists an affine open  $V = \text{Spec}(B) \subset T$  s.t. there exists an affine  $U = \text{Spec}(A) \subset X$  containing  $s$  and  $f(Y \cap U) \subset V, g(Z \cap U) \subset V$ . Then given morphisms  $B \rightarrow A/I$  and  $B \rightarrow A/J$  agree as morphisms to  $A/(I+J)$ . By the exact sequence there exists a unique morphism  $B \rightarrow A/I \cap J$  as desired.  $\square$

### 3 SCHEME THEORETIC SUPPORT

The support of a quasi-coherent sheaf may not be closed. But it's always closed under specialization.

**Lemma 3.** Let  $X$  be a scheme and  $\mathcal{F} \in \text{QCoh}(\mathcal{O}_X)$ . The support of  $\mathcal{F}$  is closed under specialization.

*Proof.* If  $x' \rightsquigarrow x$  is a specialization. Then  $\mathcal{F}_{x'}$  is a localization of  $\mathcal{F}_x$ . Hence the conclusion holds.  $\square$

But if the quasi-coherent module is of finite type, Then the support of it must be closed.

**Lemma 4.** Let  $\mathcal{F}$  be a finite type (locally finitely generated) quasi-coherent module on a scheme  $X$ . Then the support of  $\mathcal{F}$  is closed. And for  $x \in X$  we have

$$x \in \text{Supp}(\mathcal{F}) \Leftrightarrow \mathcal{F}_x \neq 0 \Leftrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0$$

*Proof.* The support of a local section is always closed in its corresponding open subset. Hence the first conclusion is obvious. The second conclusion follows from the Nakayama's lemma.  $\square$

The property of being finite type is preserved under pullback. And the support of the pullback is exactly the inverse image of the support of the original module.

**Lemma 5.** For any morphism of schemes  $f : Y \rightarrow X$  the pullback  $f^*\mathcal{F}$  is of finite type and we have

$$\text{Supp}(f^*\mathcal{F}) = f^{-1}(\text{Supp}(\mathcal{F}))$$

*Proof.* By the definition of  $f^*$ , the first conclusion holds obviously. Recall that

$$(f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y}$$

Hence  $(f^*\mathcal{F})_y \otimes \kappa(y) = \mathcal{F}_x \otimes \kappa(x) \otimes_{\kappa(x)} \kappa(y)$ . Hence  $(f^*\mathcal{F})_y \otimes \kappa(y) \neq 0$  iff  $\mathcal{F}_x \otimes \kappa(x)$  is nonzero. Hence it implies that  $x \in \text{Supp}(\mathcal{F})$  iff  $y \in \text{Supp}(f^*\mathcal{F})$ .  $\square$

Before we define the scheme theoretic support, we need the following lemma:

**Lemma 6.** *Let  $i : Z \rightarrow X$  be a closed immersion of schemes. Let  $\mathcal{I} \subset \mathcal{O}_X$  be the quasi-coherent sheaf of ideals corresponding to  $Z$ . The functor*

$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$

*is exact, fully faithful, with essential image those quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{G}$  s.t.  $\mathcal{I}\mathcal{G} = 0$ .*

*Proof.* A closed immersion is both separated and quasi-compact. Hence  $i_*$  maps quasi-coherent  $\mathcal{O}_Z$ -module to quasi-coherent  $\mathcal{O}_X$ -module. The exactness can be checked on the stalks.

Then we show the essential image is exactly that described in the lemma. We have  $\mathcal{I}(i_*\mathcal{F}) = 0$  for any quasi-coherent  $\mathcal{O}_Z$  module  $\mathcal{F}$ . Hence it suffice to show that both of the canonical map are isomorphisms

$$\begin{aligned} \mathcal{G} &\rightarrow i_*i^*\mathcal{G} \\ i^*i_*\mathcal{F} &\rightarrow \mathcal{F} \end{aligned}$$

where  $\mathcal{G}$  is a quasi-coherent  $\mathcal{O}_X$ -module s.t.  $\mathcal{I}\mathcal{G} = 0$  and  $\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_Z$ -module. Both of the isomorphisms are direct corollary of the following algebraic statement: Given a ring  $R$  and an ideal  $I$  and an  $R$ -module  $N$  s.t.  $IN = 0$ . Then the canonical map

$$N \rightarrow N \otimes_R R/I$$

is an isomorphism of  $R$ -module. □

Now we turn to consider the scheme theoretic support of a finite type quasi-coherent module.

**Definition 3** (Scheme Theoretic Support). Let  $\mathcal{F}$  be a finite type quasi-coherent module on  $X$ . The **scheme theoretic support** (S.T.S in brief) is the minimal closed subscheme  $i : Z \rightarrow X$  s.t. there exists a quasi-coherent  $\mathcal{O}_Z$ -module  $\mathcal{G}$  with  $i_*\mathcal{G} \simeq \mathcal{F}$ .

**Proposition 1.** *The scheme theoretic support always exists. And it satisfies the following properties:*

1. *If  $Spec(A) \subset X$  is any affine open and  $\mathcal{F}|_{Spec(A)} = \tilde{M}$ , then  $Z \cap Spec(A) = Spec(A/I)$ , where  $I = Ann_A(M)$ .*
2. *The quasi-coherent sheaf  $\mathcal{G}$  is unique up to unique isomorphism.*
3. *The quasi-coherent sheaf  $\mathcal{G}$  is of finite type.*
4. *The support of  $\mathcal{G}$  and of  $\mathcal{F}$  is  $Z$ .*

*Proof.* We define  $Z$  by the first property since  $Ann_A(M)_f = Ann_{A_f}(M_f)$ . By the previous lemma we see that there exists a unique quasi-coherent sheaf  $\mathcal{G}$  on  $Z$  s.t.  $\mathcal{F} \simeq i_*\mathcal{G}$ . Also,  $\mathcal{G}$  is of finite type since such a finite  $R$ -module is also a finite  $R/I$ -module. The last assertion is trivial. □

## 4 SCHEME THEORETIC IMAGE

**Definition 4** (Scheme Theoretic Image). Let  $X \rightarrow Y$  be a morphism of schemes. The **scheme theoretic image** of  $f$  is the smallest closed subscheme  $Z \subset Y$  through which  $f$  factors.

Then we show that the scheme theoretic image is always exists. But before the proof we need a lemma.

**Lemma 7.** *Let  $X$  be a scheme and  $\mathcal{F}$  a quasi-coherent  $\mathcal{O}_X$ -module. Let  $\mathcal{G} \subset \mathcal{F}$  be a submodule. There exists a unique quasi-coherent submodule  $\mathcal{G}' \subset \mathcal{G}$  s.t. For every quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{H}$*

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

*is bijective. In particular,  $\mathcal{G}'$  is the maximal quasi-coherent  $\mathcal{O}_X$ -submodule of  $\mathcal{G}$ .*

*Proof.* Let  $\{G_a\}_{a \in A}$  be the set of all the quasi-coherent submodule of  $\mathcal{G}$ . Let

$$\mathcal{G}' = \text{Image}(\bigoplus_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F})$$

Since the image of a morphisms between quasi-coherent sheaves is quasi-coherent.  $\mathcal{G}'$  is of course the largest quasi-coherent submodule of  $\mathcal{G}$ .

Let  $f : \mathcal{H} \rightarrow \mathcal{G}$  be an  $\mathcal{O}_X$ -module morphism. The image of  $\mathcal{H} \rightarrow \mathcal{G} \rightarrow \mathcal{F}$  is quasi-coherent. Hence it's contained in  $\mathcal{G}'$ . Thus the formula holds.  $\square$

*Remark.* Let  $i : Z \rightarrow X$  be a closed immersion of schemes. There is a functor

$$\bigoplus_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

defined by  $i^! \mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$  is right adjoint to  $i_*$ , where  $\mathcal{H}_Z(\mathcal{G})$  are the subsheaf generated by the local sections annihilated by  $\mathcal{I}$ .

**Lemma 8.** *For any  $f$ , the scheme theoretic image always exists.*

*Proof.* Let  $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X)$ . There exists a maximal quasi-coherent sheaf of ideals  $\mathcal{I}' \subset \mathcal{I}$ . Hence we define  $Z$  to be the closed subscheme corresponding to the ideal sheaf  $\mathcal{I}'$ . It's obvious the closed subscheme as desired.  $\square$

*Remark.* 1.  $\overline{f(X)}$  may not equal to the underlying set of the theoretic image of  $f$ .

2. The construction of the scheme theoretic image does not commute with restriction to open subschemes of  $Y$ . In other words, suppose  $f(X) \subset U \subset V$ , where  $U, V \subset X$  are open subschemes. Let the scheme theoretic images of  $f_1 : X \rightarrow V$  and  $f_2 : X \rightarrow U$  be  $Z_1$  and  $Z_2$ . Then it might happen that  $Z_2 \cap U \neq Z_1$ .

But if  $f$  is quasi-compact, things will be very awesome.

**Proposition 2.** *Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $Z \subset Y$  be the scheme theoretic image of  $f$ . If  $f$  is quasi-compact then*

1.  $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X)$  is quasi-coherent.
2.  $Z$  is the closed subscheme determined by  $\mathcal{I}$ .
3. For any open  $U \subset Y$ , the scheme theoretic image of  $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$  is equal to  $Z \cap U$ .
4.  $\overline{f(X)} = Z$ .

*Proof.* The fourth assertion follows from the third one. And both the second and the third conclusions follows from the first one. Since the property of being quasi-coherent is local. We may assume  $Y$  is affine. Since  $f$  is quasi-compact, we can decompose  $X$  into finitely many affine opens  $X = \bigcup_{i=1, \dots, n} U_i$ . Let  $X' = \coprod U_i$ , which is affine. Let  $f'$  be the composition of

$$X' \rightarrow X \rightarrow Y$$

Hence we have  $f_* \mathcal{O}_X = f'_* \mathcal{O}_{X'}$  and thus  $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \rightarrow \mathcal{O}_{X'})$ . Therefore  $\mathcal{I}$  is quasi-coherent.  $\square$

More precisely, we only need to adds points which are specializations of points in  $f(X)$  to get the scheme theoretic image if  $f$  is quasi-compact. We can use the method of valuation to show the conclusion.

**Lemma 9.** *Let  $f : X \rightarrow Y$  be quasi-compact morphism and  $Z$  the scheme theoretic image of  $f$ . Let  $z \in Z$ . There exists a valuation ring  $A$  with fraction field  $K$  and*

$$\begin{array}{ccccc} \text{Spec}(K) & \longrightarrow & X & & \\ \downarrow & & \swarrow & \downarrow & \\ \text{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

s.t. the closed point of  $\text{Spec}(A)$  maps to  $z$ . In particular, every point of  $Z$  is the specialization of a point of  $f(X)$ .

*Proof.* Let  $z \in \text{Spec}(R) = V \subset Y$  be an affine open. And  $z$  corresponds to the prime  $\mathfrak{p} \subset R$ . The intersection  $Z \cap V$  is the scheme theoretic image of  $f^{-1}(V) \rightarrow V$ . Hence WLOG we assume that  $Y = \text{Spec}(R)$ . Let  $X = \bigcup_{i=1,\dots,n} U_i$  be a finite affine open covering, where  $U_i = \text{Spec}(R)$ . Let  $I = \text{Ker}(R \rightarrow A_1 \times \dots \times A_n)$ . Hence  $Z$  corresponds to the closed subscheme  $\text{Spec}(R/I)$ .

Now we only need to find a prime  $\mathfrak{p}_i \subset \mathfrak{p}$  and some prime  $\mathfrak{q}_i \subset A_i$  lying over  $\mathfrak{p}_i$ . Then we can choose a valuation ring  $A \subset K = \kappa(q_i)$  dominating the local ring  $R_{\mathfrak{p}}/\mathfrak{p}_i R_{\mathfrak{p}} \subset \kappa(p_i) \subset \kappa(q_i)$ .

Since we have  $I_{\mathfrak{p}} \subsetneq R_{\mathfrak{p}}$  because  $I \subset \mathfrak{p}$ . We see that  $R_{\mathfrak{p}} \rightarrow (A_1)_{\mathfrak{p}} \times \dots \times (A_n)_{\mathfrak{p}}$  is not zero. Hence one of the rings  $(A_i)_{\mathfrak{p}}$  is not zero. And there exists an  $i$  and a prime  $\mathfrak{q} \subset A_i$  lying over a prime  $\mathfrak{p}_i \subset \mathfrak{p}$ .  $\square$

Now let us consider the comma category of morphisms of schemes. The objects of such a category are morphisms of schemes  $f : X \rightarrow Y$ . The morphisms between  $f_1 : X_1 \rightarrow Y_1$  and  $f_2 : X_2 \rightarrow Y_2$  is a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow & & \downarrow \\ X_2 & \xrightarrow{f_2} & Y_2 \end{array}$$

Taking scheme theoretic image is an functor from the category of morphisms of schemes to the category of schemes in fact. In other words, we have the following lemma.

**Lemma 10.** *Let*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ g_1 \downarrow & & \downarrow g_2 \\ X_2 & \xrightarrow{f_2} & Y_2 \end{array}$$

be a commutative diagram of schemes. Let  $Z_i \subset Y_i$  be the scheme theoretic image of  $f_i$ . Then the morphism  $Y_1 \rightarrow Y_2$  induces a canonical morphism  $Z_1 \rightarrow Z_2$  and a commutative diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & Z_1 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow & & \downarrow \\ X_2 & \longrightarrow & Z_2 & \longrightarrow & Y_2 \end{array}$$

*Proof.* The scheme theoretic image of  $Z_2$  in  $Y_1$  is a closed subscheme of  $Y_1$  through which  $f_1$  factors. In other words, the scheme theoretic image of  $g_2 \circ f_1$ , denoted by  $Z'$ , is contained in the scheme theoretic image of  $f_2$ . And of course  $f_1$  factors through the scheme theoretic inverse image of  $Z'$  by  $g_2$ .  $\square$

Suppose  $f : X \rightarrow Y$  is a morphism from a reduced scheme  $X$ . Then the scheme theoretic image of  $f$  is also the same case as  $f$  is quasi-compact.

**Lemma 11.** *Let  $f : X \rightarrow Y$  be a morphism of schemes. If  $X$  is reduced, then the scheme theoretic image of  $f$  is the reduced scheme theoretic structure on  $\overline{f(X)}$*

It's a direct corollary of the following lemma:

**Lemma 12.** *Let  $X$  be a scheme and  $Z \subset X$  be a closed subscheme. Let  $Y$  be a reduced scheme. A morphism  $f : Y \rightarrow X$  factors through  $Z$  iff  $f(Y) \subset Z$  (set theoretically).*

## 5 SCHEME THEORETIC CLOSURE AND SCHEME THEORETIC DENSE

At first, we consider an immersion  $h : Z \rightarrow X$ . The most interesting cases are  $h$  being quasi-compact or  $Z$  is reduced.

**Proposition 3.** Let  $h : Z \rightarrow X$  be an immersion. If  $h$  is quasi-compact, or  $Z$  is reduced, then we can factor  $h = i \circ j$  with  $j : Z \rightarrow \bar{Z}$  an open immersion and  $i : \bar{Z} \rightarrow X$  a closed immersion.

*Proof.* 1. Suppose  $h$  is quasi-compact. Since  $h$  is an immersion, it's also quasi-separated. Hence  $h_* \mathcal{O}_Z$  is a quasi-coherent sheaf of  $\mathcal{O}_X$ -module. And  $\mathcal{I} = \text{Ker}(\mathcal{O}_X \rightarrow h_* \mathcal{O}_Z)$  is quasi-coherent. Hence let  $\bar{Z} \subset X$  be the closed subscheme corresponding to  $\mathcal{I}$ . Then  $h$  obviously factor through  $i : \bar{Z} \rightarrow X$  which is a closed immersion. To see  $j : Z \rightarrow \bar{Z}$  is an open immersion, let  $U \subset X$  be an open subscheme s.t.  $h$  induces a closed immersion of  $Z$  into  $U$ . Then it's clear that  $\mathcal{I}|_U$  is the sheaf of ideal corresponding to the closed immersion  $Z \rightarrow U$ .

2. Suppose  $Z$  is reduced. The assertion is a direct conclusion of *Lemma 12*. □

Then we define the scheme theoretic closure for general open subschemes.

**Definition 5.** Let  $X$  be a scheme and  $U \subset X$  be an open subscheme.

1. The **scheme theoretic closure** (S.T.C in brief) of  $U$  is the scheme theoretic image of  $i : U \hookrightarrow X$ .
2. We say  $U$  is **scheme theoretically dense** (S.T.D in brief) in  $X$  if for every open  $V \subset X$  the S.T.C of  $U \cap V$  in  $V$  is equal to  $V$ .

*Remark.* In general, the S.T.C of  $U$  being  $X$  does not imply  $U$  is S.T.D. in  $X$ . But if  $i : U \rightarrow X$  is quasi-compact, then  $U$  is S.T.D. in  $X$  iff the S.T.C of  $U$  is  $X$ .

There is an criterion for being S.T.D:

**Proposition 4.** Let  $j : U \rightarrow X$  be an open immersion of schemes. Then  $U$  is scheme theoretically dense in  $X$  iff  $\mathcal{O}_X \rightarrow j_* \mathcal{O}_U$  is injective.

*Proof.* If  $\mathcal{O}_X \rightarrow j_* \mathcal{O}_U$  is injective, then the same is true when restricted to any open  $V$  of  $X$ . Hence the scheme theoretic closure of  $U \cap V$  in  $V$  is equal to  $V$ . Conversely, suppose  $\mathcal{O}_X \rightarrow j_* \mathcal{O}_U$  is not injective. Then we can find an affine open  $\text{Spec}(A) = V \subset X$  and a nonzero element  $f \in A$  s.t.  $f$  maps to zero in  $\gamma(V \cap U, \mathcal{O}_X)$ . Hence the scheme theoretic closure of  $V \cap U$  in  $V$  is contained in  $\text{Spec}(A/(f))$ . □

The intersection of two S.T.D open subschemes is also S.T.D.

**Lemma 13.** Let  $U, V$  be S.T.D open subschemes of  $X$ , then  $U \cap V$  is S.T.D in  $X$ .

*Proof.* Let  $W \subset X$  be any open. The composition of the morphisms  $\mathcal{O}_X(W) \rightarrow \mathcal{O}_X(W \cap V) \rightarrow \mathcal{O}_X(W \cap V \cap U)$  is injective. □

Then we return to the case  $h : Z \rightarrow X$  be an immersion, where  $h$  is quasi-compact or  $Z$  is reduced.

**Lemma 14.** Let  $h : Z \rightarrow X$  be an immersion. Assume  $h$  is quasi-compact or  $Z$  is reduced. Let  $\bar{Z} \rightarrow X$  be the scheme theoretic image of  $h$ . Then  $Z \rightarrow \bar{Z}$  is an open immersion which identifies  $Z$  with a S.T.D open subscheme of  $\bar{Z}$ . And  $Z$  is topologically dense in  $\bar{Z}$ .

*Proof.* By *Proposition 3*, the underlying set of  $Z$  is exactly the topological closure of  $Z$ . Furthermore, if  $Z$  is reduced, then the theoretic image has also the unique reduced structure on  $\bar{Z}$ . And if  $h$  is quasi-compact, then  $\mathcal{O}_{\bar{Z}} \rightarrow i_* \mathcal{O}_Z$  is an injection. □

Thus in reduced scheme, an open subscheme is topologically dense is exactly the same as S.T.D.

**Proposition 5.** Let  $X$  be a reduced scheme and  $U \subset X$  be an open subscheme. TFAE

1.  $U$  is topological dense in  $X$
2. the S.T.C of  $U$  in  $X$  is  $X$

3.  $U$  is S.T.D in  $X$

Conversely, we have

**Lemma 15.** *Let  $X$  be a scheme and  $U \subset X$  be a reduced open subscheme. TFAE*

1. The S.T.C. of  $U$  in  $X$  is  $X$
2.  $U$  is S.T.D. in  $X$

*If this holds then  $X$  is also reduced.*

At last, we consider the equalizer of morphisms in the category of schemes.

**Proposition 6.** *Let  $X, Y$  be schemes over  $S$  and  $a, b : X \rightarrow Y$  be morphisms of schemes over  $S$ . There exists a largest locally closed subscheme  $Z \subset X$  s.t.  $a|_Z = b|_Z$ , namely the equalizer of  $(a, b)$ . If  $Y$  is separated over  $S$ , then  $Z$  is a closed subscheme.*

*Proof.* The equalizer of  $(a, b)$  is for categorical reasons the fibre product  $Z$  in the following diagram

$$\begin{array}{ccc} Z = Y \times_{(Y \times_S Y)} X & \longrightarrow & X \\ \downarrow & & \downarrow (a, b) \\ Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y \end{array}$$

Since being a (closed) immersion is preserved under base change. The proposition follows.  $\square$

Two continuous maps that agree on a dense open subset is equal. It has similar generalization in the category of schemes.

**Lemma 16.** *Let  $S$  be a scheme and  $X, Y$  be schemes over  $S$ . Let  $f, g : X \rightarrow Y$  be morphisms of schemes over  $S$ . Let  $U \subset X$  be an open subscheme s.t.  $f|_U = g|_U$ . If the S.T.C of  $U$  is  $X$  and  $Y \rightarrow X$  is separated, then  $f = g$ .*